Lecture notes

Macroeconomics
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1 Growth in Transition: Some facts and definitions

Transition has been characterized by a U-shaped behavior of output. Fisher et al. have suggested to think in terms of a two-component dynamics: one, transitional behavior of output, the second traditional growth (for instance of the Solow-type). Aggregate dynamics resulted from the evolution of the two components with traditional growth theories becoming more and more relevant over time. The weight of the two components changed over time in a asymmetric way across countries (Central Eastern Europe and FSU, for instance).

With respect to the period of central planning, transition can be seen as a switch of regime, causing a shift in the trend growth rate of the economies. Transition can be considered as a path joiming such two different trends.

A possible interpretation is to consider the growth process in transition in a way that is similar to the approach followed by Parente and Prescott (PP) (2004). The state sectors inherited from the previous regime can be considered as the Malthus sector in PP, with a stagnant dynamics of total factor productivity.

Parente and Prescott (2004) make tthe following points:

1. Identify *modern economic growth* (term introduced by Kuznets, an economic historian): sustained growth in incomes per capita. Before 1700 incomes per capita did not grow on a sustained basis (Malthus regime). As a consequence, differences in incomes per capita across countries were persistent, but were not very large. England was the first country to enter modern economic growth, characterized by sustained growth in incomes per capita. To appreciate the extraordinary performance observed after Wrold War II, it suffices to notice that starting in 1750 it took England 100 years to double its income per capita level, while in the 20th century Western European countries doubled their incomes per capita on average every 35 years.

2. Countries entered modern economic growth at different points in time. In Africa, for instance, such period has yet to start. In several cases countries

entered their period of modern economic growth more than 100 years after the leader (England).

3. Some countries have reduced over time the gap in their income capita with respect to the leader (convergence phenomenon). However, others have not done it (for instance the gap between Latin America and the leader of the last century (the USA) has remaine constant for the last 100 years.

The socialist period could be considered a period of missed opportunity for moving the economies to the *modern growth regime*.

Economic theory should be able to account for such facts.

2 The Solow model

Production: output is produced using labor and capital.

$Y(t) = F(K(t), L(t))$

The saving rate (the share of output saved) is assumed constant:

$s(.) = s > 0$

The capital stock increases over time through investment, net of depreciation

$\frac{dK}{dt} = \dot{K} = sF(K, L) - \delta K$

Assumptions on $F(.)$

$\frac{\partial F}{\partial K} > 0 \quad \frac{\partial^2 F}{\partial K^2} < 0$

$\frac{\partial F}{\partial L} > 0 \quad \frac{\partial^2 F}{\partial L^2} < 0$

Constant return to scale

$F(\lambda K, \lambda L) = \lambda F(K, L)$

Inada conditions

$lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty$

$lim_{K\to\infty} F_K = \lim_{L\to\infty} F_L = 0$

Constant return to scale allow us to rewrite the production function in terms of per capita variables

$Y = F(K, L) = Lf(K/L, 1) = Lf(k)$

with

$k = K/L \quad \text{and} \quad y = f(k)$

$\partial Y/\partial K = f'(k)$

$\partial Y/\partial L = f(k) - kf'(k)$

Assume population grows at the constant term n

$\frac{dL}{dt} * \frac{1}{L} = n$

dividing the investment equation by L

$\frac{\dot{K}}{L} = sf(k) - \delta k$

$\dot{k} = \left(\frac{\dot{K}}{L}\right) = \frac{\dot{K}}{L} - nk \quad \text{note:} \;\; d(K/L)/dt = \left(\frac{\dot{K}}{L}\right) = \frac{(dK/dt)*L-(dL/dt)K}{L^2}$

$\dot{k} = sf(k) - (n + \delta)k$

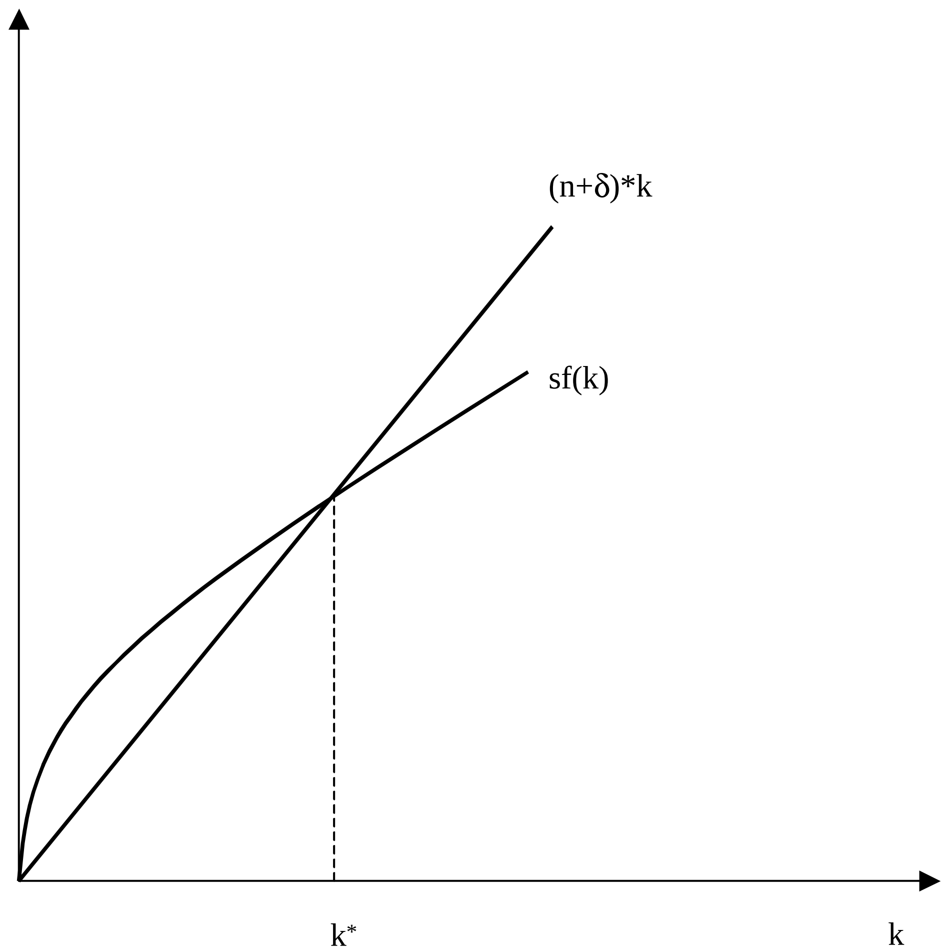
Steady state:

$\dot{k} = 0$
$sf(k) = (n + \delta)k$

solve for steady-state $k = k^*$

$y^* = f(k^*)$

steady state consumption $c^* = f(k^*) - (n + \delta)k^*$


(n+δ)*k
sf(k)
k*
k

2.1

2.2 Golden rule and dynamic inefficiency

Optimal level of steady state consumption

Choice variable k^*
$max_{k^*} \qquad c^* = f(k^*) - (n+\delta)k^*$

First order condition

$f'(k^*) - (n+\delta) = 0$

or

$f'(k^*) = (n+\delta)$

gives the steady state stock of capital that maximizes consumption, thus utility. This is not necessarily the capital stock obtained in steady state for a given saving rate.

When $k^* > k_{gold}$ the economy displays dinamic inefficiency: it saves and invest too much (case of socialist economies for instance).

2.3 Dynamics

Re-write the steady state relationship by dividing both terms by k

Thus steady state occurs when

$s\frac{f(k)}{k} = n + \delta$

Now, the left-hand side declines with k

$\partial\left(s\frac{f(k)}{k}\right)/\partial k = \frac{f'(k)k - f(k)}{k^2} < 0 \quad \text{as} \qquad f'(k) < f(k)/k$

Note that $f''(k) < 0$ means that the marginal product of capital declines. This implies that the marginal product of capital is smaller than the average product of capital. Note also that as long as $L < \infty$ the marginal product of labor is positive:

$\partial Y/\partial L = f(k) - f'(k)k > 0$, which again confirms our result above.

Note that at the left of the steady state $sf(k)/k > n + \delta$

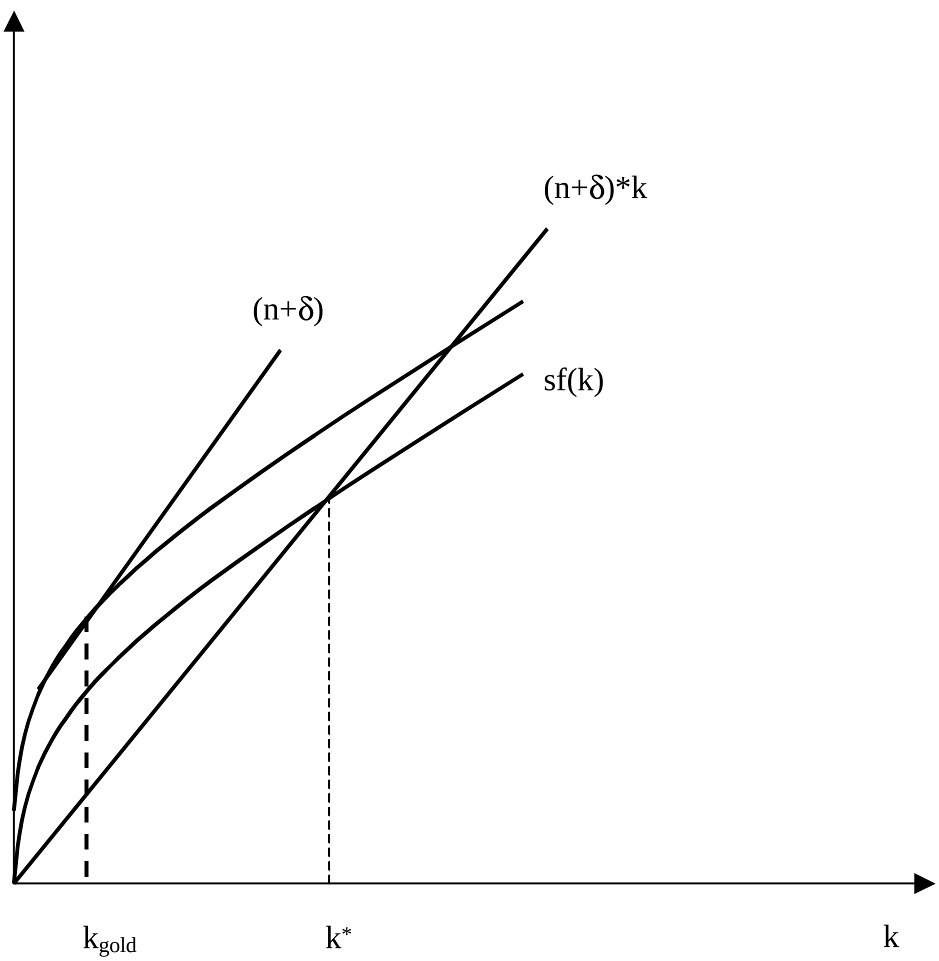
From the equation of motion of k this implies that k is increasing $(\dot{k} > 0)$, and the opposite is true when k is above its steady state value.

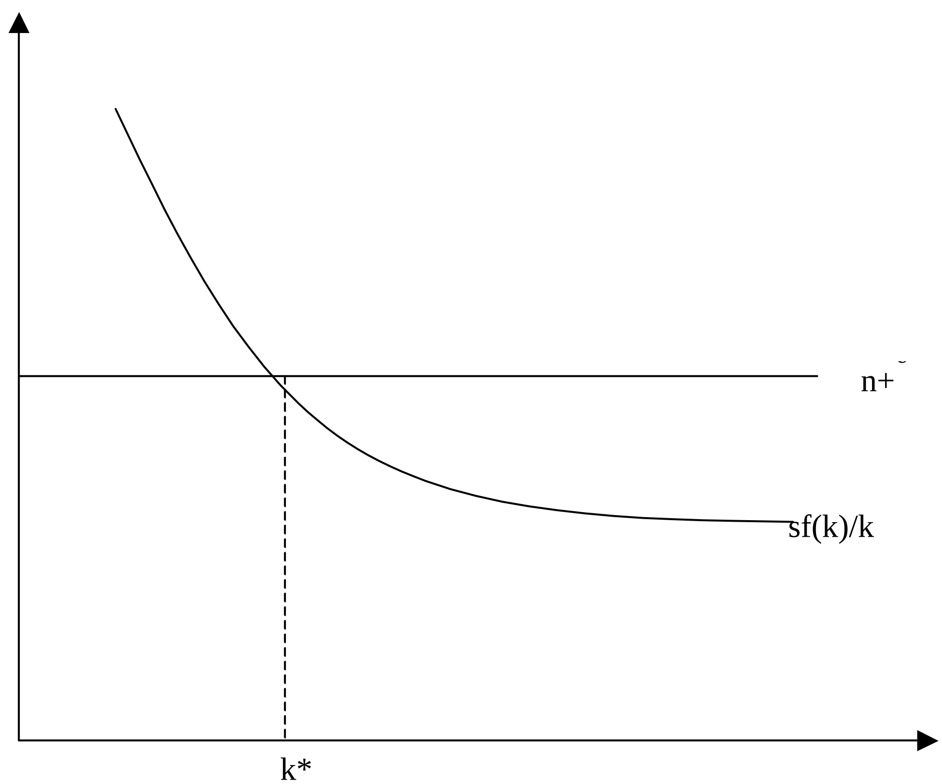
The growth rate of output per capita along the transition is

$\gamma_y = \frac{dy}{dt} * \frac{1}{y} = f'(k) * \dot{k}/f(k) = [kf'(k)/f(k)] * \gamma_k$

where $\gamma_k = \frac{\dot{k}}{k}$

Thus the rate of growth of output is an increasing function of the rate of growth of per capita capital.


(n+δ)*k
(n+δ)
sf(k)
k_gold
k*
k


n+
sf(k)/k
k*

2.4 Convergence in the Solow model

$\partial\gamma_k/\partial k = s(f'(k) - f(k)/k)/k < 0$

This means that a country with a lower per capita stock of capital grows faster than a country with a higher stock of per capita capital. Given the production function, this means that a poorer country (lower output per capita) grows faster than a richer country. Thus, the Solow model predicts absolute convergence. However, we can also analyze conditional convergence. Assume, for instance, that a richer country has a higher saving ratio than a poor country. This means that the two countries have different steady states. At one point in time it is possible that the rich country grows faster than a poor country. This occurs if the rich country is further away from its steady state, while the poor country is closer to its own steady state. Consistent with this view is the theory of convergence clubs.

3 Technological progress in the Solow model.

Labor-augmenting technological progress

Production is now:

$Y(t) = F(K(t), L(t) * A(t))$

with $\frac{dA}{dt} = \dot{A}/A = x > 0$

$\gamma_k = sF\left[k, A(t)\right]/k - (n + \delta)$

Because of constant returns to scale

$F\left[k, A(t)\right]/k = F\left[1, A(t)/k\right]$

Steady state growth: constant rate of growth

As s, n and δ are constant parameters, constant rates of growth requires that $A(t)$ grows at the same rate as k. Thus, in steady state

$\dot{k}/k = x$

$\gamma_k = \gamma_k^* = x$

The difference with the Solow model without technological change is that at steady state per capita capital and per capita output are not constant but they grow at a constant rate. Redefine variables in terms of A(t):

$\widehat{k} = k/A(t), \ \ \widehat{y} = f(\widehat{k})$

Note that $\widehat{k} = K/LA(t)$

$LA(t) =$ effective labor, that is the amount of labor multiplied by its efficiency

$$\frac{d}{dt}\left(\widehat{k}\right) = \frac{\dot{K}*L*A(t) - \dot{L}*A(t)*K - \dot{A(t)}*L*K}{(LA(t))^2} = \frac{\dot{K}}{(LA(t))} - n*\frac{K}{LA(t)} - x*\frac{K}{LA(t)}$$

Using the equation for $\dot{K}$

$$\frac{d}{dt}\left(\widehat{k}\right) = \left(sF(K, L*A(t) - \delta K\right)/L*A(t) - n*\widehat{k} - x*\widehat{k} = sf(\widehat{k}) - \delta*\widehat{k} - x*\widehat{k}$$

Thus, the rate of growth of the new definition of capital is

$$\gamma_{\widehat{k}} = \frac{d}{dt}(\widehat{k})/\widehat{k} = \frac{d}{dt}(K/LA(t))/\widehat{k} = \left[\frac{\dot{K}*L*A(t) - \dot{L}*A(t)*K - \dot{A(t)}*L*K}{(LA(t))^2}\right]*\frac{1}{\widehat{k}} = \left[sf(\widehat{k}) - \delta*\widehat{k} - n*\widehat{k} - x*\widehat{k}\right]*\frac{1}{\widehat{k}}$$

or

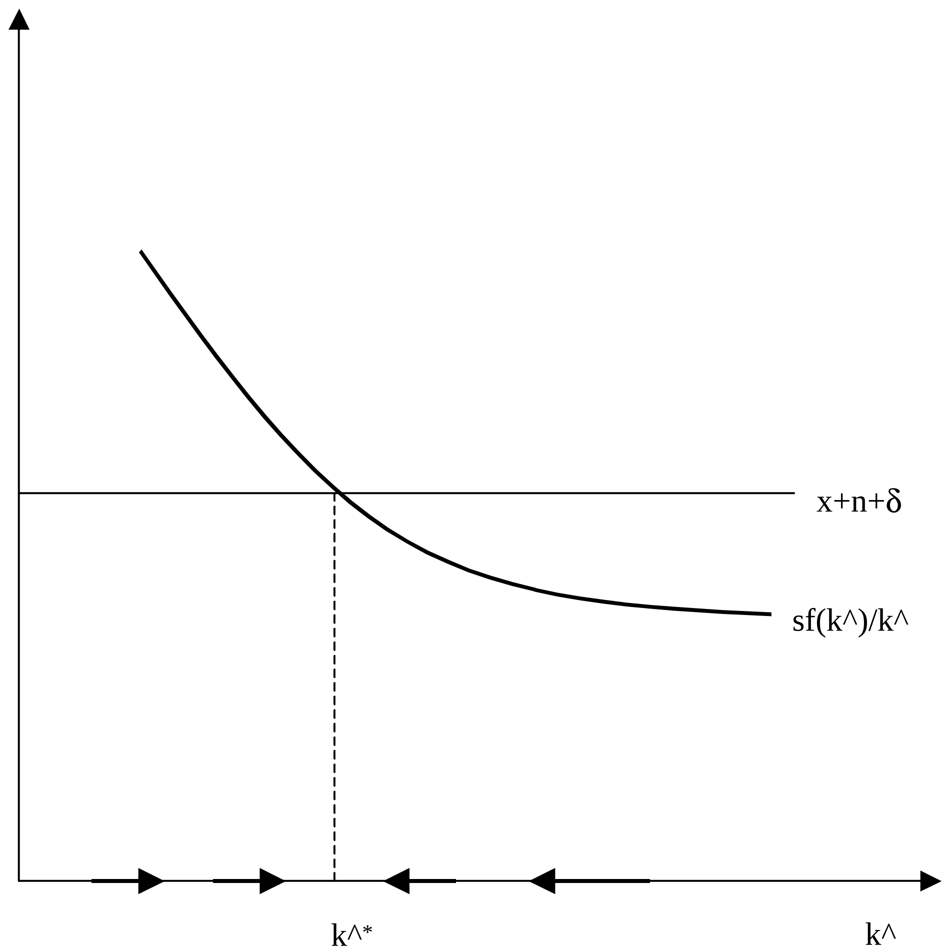
$$\gamma_{\widehat{k}} = s*f(\widehat{k})/\widehat{k} - \delta - n - x$$

Steady state is characterized by

$$\gamma_{\widehat{k}} = 0$$

$$s*f(\widehat{k}^*)/\widehat{k}^* = \delta + n + x$$

The properties of this model are analogous to the simple Solow model, once variables are re-interpreted in terms of efficiency values. The difference is that in steady states there are different, constant, rates of growth of per capita capital and output, and not only different levels. Note, that long run growth depends only on exogenous factors.


x+n+δ
sf(k^)/k^
k^*
k^

4 A simple model of endogenous growth

AK-model.

Production

$Y = AK$

Main difference, there are no diminishing returns. Lack of realism? No if we consider K as including both physical and human capital.

Per capita

$y = A * k$

from which

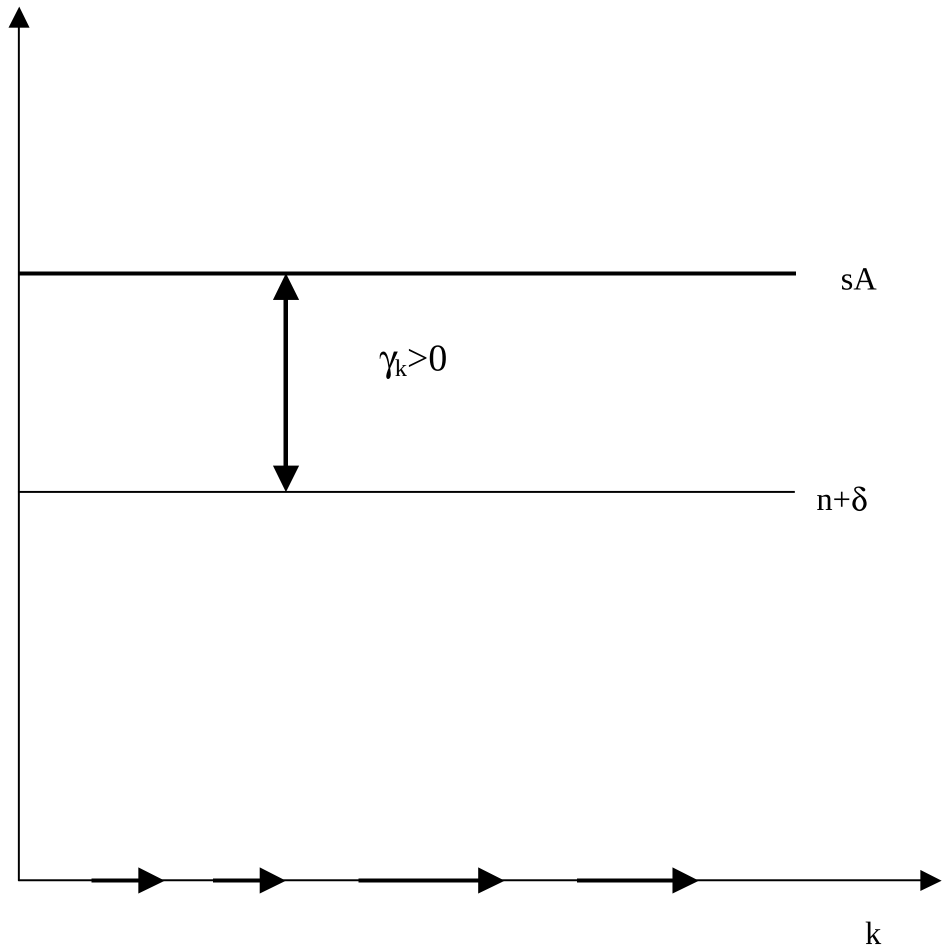
$f(k)/k = A$

Thus,

$\gamma_k = sA - (n + \delta)$

From Figure 5, we can see that there is a permanent rate of growth of per capita capital and thus per capita output, as long as

$sA > (n + \delta)$



γ_k is the vertical distance between sA and $(n+\delta)$

The model does not predict either absolute (which seems consistent with data) or conditional convergence (at odds with data).

To obtain conditional convergence in an endogenous growth model we can modify the production function in the following way:

$$Y = AK + BK^{\alpha}L^{1-\alpha}$$

In per capita terms

$$y = Ak + Bk^{\alpha}$$

$$f(k)/k \to A \quad \text{as} \quad k \to \infty$$

$$sf(k)/k \to sA \quad \text{as} \quad k \to \infty$$

The steady state rate of growth is constant and equal to $sA - (n+\delta)$

while γ_k is higher the smaller is $k(0)$

5 Adding Human capital to the Solow model

$Y_t = K_t^{\alpha}(A_t H_t)^{1-\alpha}$

Accumulation of physical capital:

$\dot{K}_t = sY_t - \delta K_t$

dynamics of technology:

$\dot{A}_t = gA_t$

Human capital is accumulated by using human resources (student time):

$H_t = L_t G(E)$

where E stands for education; $G'(E) > 0$

Example: $G(E) = e^{\phi E}$ with $\phi > 0$

Define with lower case letters per capita variables in terms of efficiency and level of education:

$k = K/\left[AG(E)L\right]$

$\dot{k}_t = sf(k_t) - (n+g+\delta)k_t = sk_t^{\alpha} - (n+g+\delta)k_t$

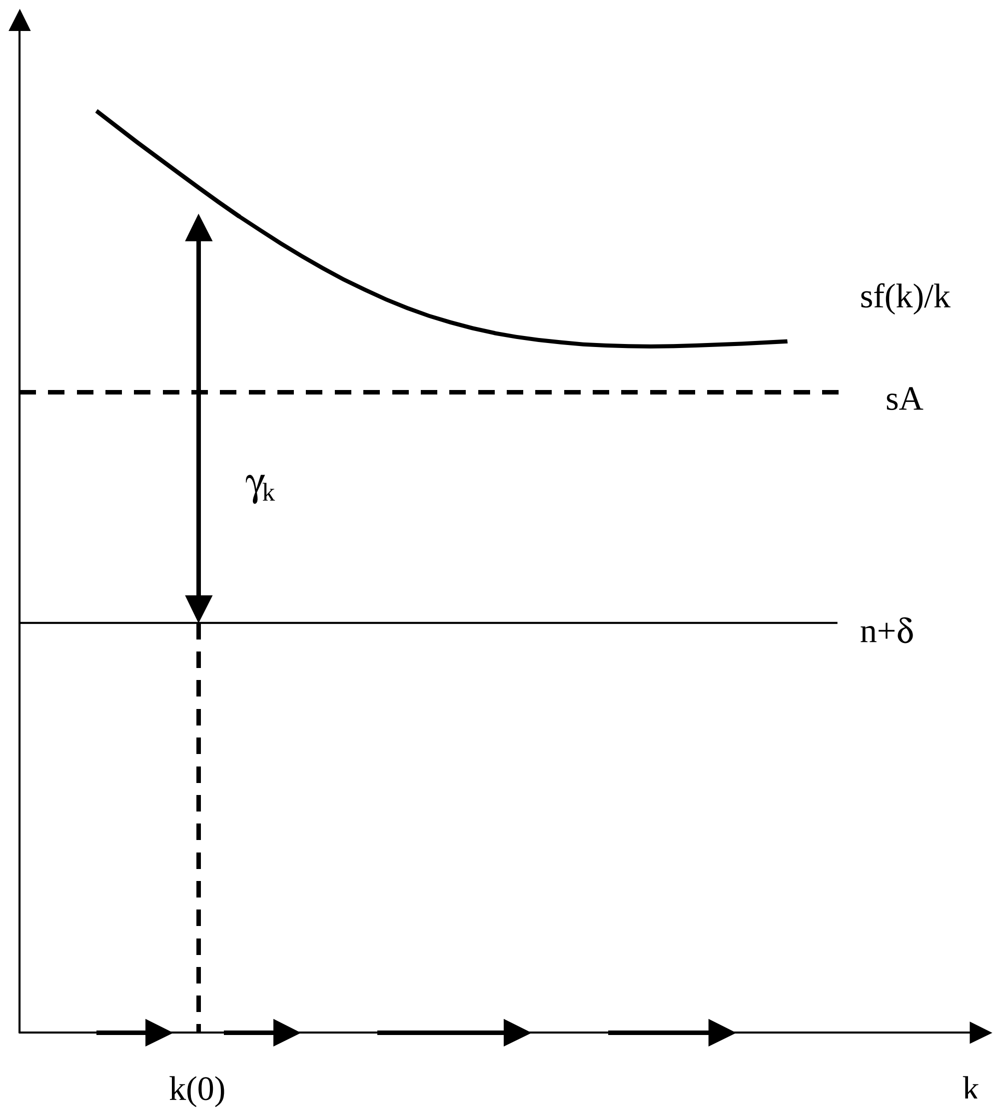
Note that the model is exactly the same as the Solow model without human capital. The difference is that the variables are now defined in terms of levels of education.

The balanced growth path is defined by

$\dot{k}_t = 0, \qquad sk^{\alpha} = (n+g+\delta)k$

which implies

$k^* = \left(\frac{s}{n+g+\delta}\right)^{1/(1-\alpha)}$


sf(k)/k
sA
γk
n+δ
k(0)
k

and the equilibrium output per worker in efficiency units is

$$y^* = \left(\frac{s}{n+g+\delta}\right)^{\alpha/(1-\alpha)}$$

Along the balanced growth path capital per worker k^* and y^* are independent of E. An increase in human capital, through an increase in years of schooling, has an effect of capital and output per worker (productivity):

$$\frac{Y}{L} = yAG(E) = yAe^{\phi E}$$

$$\partial \left(\frac{Y}{L}\right) / \partial E = yAG'(E) = \phi y A e^{\phi E}$$

Thus, when comparing countries we should expect that relative incomes per capita are affected by the levels of education. Human capital is thus one candidate variable for conditioning convergence.

6 Poverty traps

It is not difficult to generate poverty traps in the Slow model or in a simple endogenous growth model. Poverty traps are characterized by low-level of output per capita in steady states. This occurs whne the the average product of capital has the familiar negative slope at low level of k but it turns positive passed a certain critical value of k and then it slopes negatively after another critical value of k.

A common explanation has to do with the role of agriculture and industry during the phase of development. Agriculture is considered a sector where diminishing returns dominate, while industry, at least for a given period, may display increasing returns (because of technological change, human capital accumulation etc.).

7 Convergence

7.1 Some basic notions

Here we introduce some useful used in discussing growth.

Compounding: suppose a country grows at an average rate of growth of 1.2% and has an income per capita of 1348 US$. How many years it would take for this country to double its income level?

$2 * 1348 = 1348 * (1.012)^N \quad \text{or} \quad 2 = (1.012)^N$

To solve for N take the log of the above expression

$\ln 2 = N * \ln(1.012)$

or

$N = \ln 2 / \ln(1.012) = 58.108$

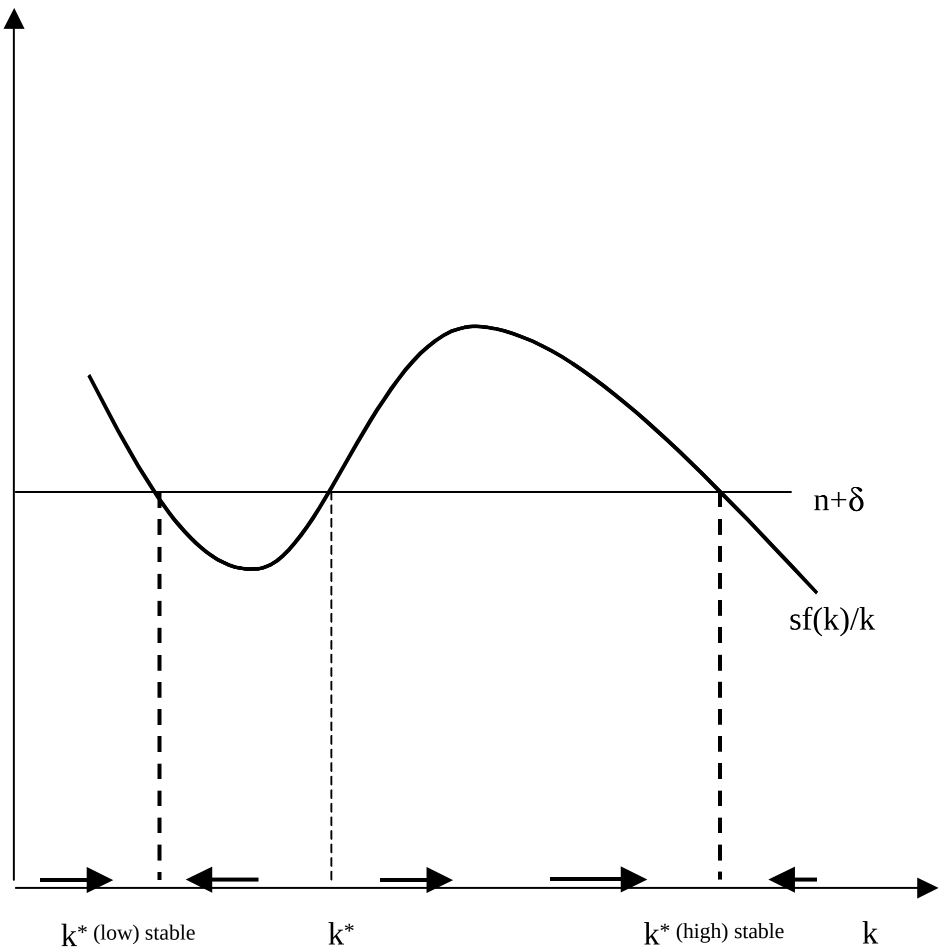
$N = \ln 2 / \ln(1.02) = 35.003$

Therefore, it would take roughly 58 years to double the level of income per capita for a country growing a 1.2% per year. A more interesting question is: if country 1, which has an income per capita of 10,010 US$, grows at a rate of 6.1% per annum while country 2, which has an income per capita of 21,558 US$, grows at 1.9% per annum, how many years would it take for country 1 to reach the same income level of country 2 (or, in other words, how many years would it take for country 1 to converge to country 2).

$21558 * (1.019)^N = 10010 * (1.061)^N$

take logs:

$\ln 21558 + N * \ln(1.019) = \ln 10010 + N * \ln(1.061)$


n+δ
sf(k)/k
k* (low) stable
k*
k* (high) stable
k

$$N*(\ln(1.061)-\ln(1.019))=\ln(21558)-\ln(10010)$$

$$N=(\ln(21558)-\ln(10010))/(\ln(1.061)-\ln(1.019))=18.994$$

Another interesting question. The Prime Minister of a country, say Slovakia, in the electoral campaign states: "by the year 2020 Slovakia will have the same income per capita as the average EU country". Suppose that in 2005 Slovakia has an income per capita of around 60% of EU average. EU is expected to grow at around 1.5% per year. What would be the yearly rate of growth of Slovakia that will make true the statement of the Prime Minister?

Note that from today to 2020 there are 15 years. Define y as income per capita; we know that $y_{2005}^{Sk}/y_{2005}^{EU}=0.6;$ thus:

$$y_{2005}^{EU}*(1.015)^{15}=y_{2005}^{Sk}*(x)^{15}$$

with $x=1+$rate of growth

$$(1.015)^{15}=\frac{y_{2005}^{Sk}}{y_{2005}^{EU}}*(x)^{15} \quad \text{or} \quad (1.015)^{15}=0.6*(x)^{15}$$

$$x^{15}=\frac{(1.015)^{15}}{0.6} \quad \text{or} \quad x=\frac{1.015}{(0.6)^{(1/15)}}=1.0502$$

Another method: take logs

$$15*\ln x=\ln(\frac{(1.015)^{15}}{0.6}) \quad \text{or} \quad \ln x=0.73415/15=4.8943\times10^{-2}=0.048943$$

take the exponential function

$$e^{\ln x}=x=e^{4.8943\times10^{-2}}=1.0502$$

which says that Slovakia's income per capita has to grow at 5.02% per year, or 3.5% faster than EU average.

7.2 Notions of convergence

$\beta-$*convegence*, or absolute convergence when poorer countries grow faster than richer ones. This implies that there exists a negative relatonship between rates of growth of income and the initial level of income;

conditional $\beta-$ *convergence*: when $\beta-$ *convergence* refers to the partial correlation between rate of growth and initial level of income, once we control for other factors affecting growth. This means that each country has its own steady state, different from other countries. The implication is that the larger is the distance from its own steady state, the faster is the rate of growth. However, this does not imply that poorer countries grow faster than richer countries;

$\sigma - convergence$ is defined by the reduction of dispersion of income per capita across countries over time (reduction of standard deviation or variance). This notion of convergence refers to the fact that the distribution of income per capita across countries become less unequal. Absolute convegence and dispersion of income across countries are related. However, even when there is absolute convergence there could be increase of income dispersion.

7.2.1 Relationship between β and $\sigma - convergence$

$\ln(y_{it}) = a + (1-b) * \ln(y_{i,t-1}) + u_{it}$

$0 < b < 1$

u_{it} is a disturbance term with zero mean and variance σ_{μ}^2

The solution of the above difference equation is

$\ln(y_{it}) = a/b + (1-b)^t * \ln(y_{i,0}) + \sum_{j=0}^{t}(1-b)^j u_{it-j}$

(solve it by bacward substitution, looking at the equation at time t-1, t-2 etc..)

Disregard for a moment the stochastic term u. In steady state

$\ln(y_{it}) = \ln(y_{i,t-1}) = \ln(y_i^*) = a/b$

with $b < 1$ the process converges to $\ln(y_i^*) = a/b$

as with $t \rightarrow \infty$ $(1-b)^t \rightarrow 0$

Define $D = \sigma_y^2$, the variance of incomes per capita, which is a measure of income dispersion

$\sigma_{y_{it}}^2 = (1-b)^2 \sigma_{y_{i,t-1}}^2 + \sigma_u^2$

or

$D_t = (1-b)^2 D_{t-1} + \sigma_u^2$

In steady state $D_t = D_{t-1} = D^*$

$D^* = \sigma_u^2/(1-(1-b)^2)$

or

$\sigma_u^2 = D^*(1-(1-b)^2)$

Substituting this value in the expression for D_t

$D_t = (1-b)^2 D_{t-1} + D^*(1-(1-b)^2) = D^* + (1-b)^2(D_{t-1} - D^*)$

As $(1-b)^2 < 1$

D converges to D^*. This means that if D starts below D^*, D grows over time (the opposite is true if D starts above D^*). In such a case, even though there is absolute convergence, as $b < 1$, income dispersion increases.

Solve

$D_t = D^* + (1-b)^2(D_{t-1} - D^*)$

by iteration

$D_{t-1} = D^* + (1-b)^2(D_{t-2} - D^*)$

and

$D_{t-2} = D^* + (1-b)^2(D_{t-3} - D^*)$

$D_{t-1} = D^* + (1-b)^2(D^* + (1-b)^2(D_{t-3} - D^*) - D^*)$

$D_t = D^* + (1-b)^{2+3}(D_{t-3} - D^*)$

Thus the general solution if we keep iterating backwords is

$D_t = D^* + (1-b)^{2t}(D_0 - D^*)$

As $t \to \infty$ $\quad D_t \to D^*$ $\quad$ as $\quad (1-b)^{2t} \to 0$

8

9 Optimal growth (optimizing households)

Inter-temporal model of optimal decision of consumption over time, thus savings. Assume infinitely-lived households that

$\max \ \sum_{t=0}^{\infty} \beta^t U(c_t)$

where $\beta < 1$ is the subjective discount factor $\beta = \frac{1}{1+\delta}$ where $\delta > 0$ is the subjective rate of time preference

subject to an inter-temporal budget constraint, that states that the present value of the whole stream of consumption is equal to the present value of the whole stream of income, with r being the interest rate (to derive the IBC one needs to impose the so-called absence of *Ponzi games*).

$\sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t} = \sum_{t=0}^{\infty} \frac{I_t}{(1+r)^t}$

Assume first that income is exogenous income and that households live for two periods. Furthermore, assume that utility is time-separaple:

$U(c_1) + \beta U(c_2)$

with $U'(c_i) > 0$ and $U''(c_i) < 0$

Budget constraints in the two periods

$c_1 = I_1 - s$
$c_2 = I_2 + s(1+r)$

where s denotes savings

the period 2 budget constraint can be rewritten as

$\frac{c_2}{(1+r)} = \frac{I_2}{(1+r)} + s$

s appears in both budget constraints (period 1 and 2), and thus represents the link between present and future:

adding the 2 budget constraints we can eliminate s to get

$c_1 + \frac{c_2}{(1+r)} = I_1 + \frac{I_2}{(1+r)}$

which is the inter-temporal budget constraint for the case of two periods.

The consumer maximizes its utility by picking c_1 and c_2.
Set up the Lagrangean:

$$L = U(c_1) + \beta U(c_2) + \lambda \left(I_1 + \frac{I_2}{(1+r)} - c_1 - \frac{c_2}{(1+r)} \right)$$

first order conditions

$$\partial L / \partial c_1 = 0 = \partial L / \partial c_2$$

$$U'(c_1) - \lambda = 0$$
$$\beta U'(c_2) - \lambda \frac{1}{(1+r)} = 0$$

or

$$\frac{U'(c_1)}{U'(c_2)} = \beta(1+r) = \frac{1+r}{1+\delta}$$

The equation above states that consumption grows over time $c_2 > c_1$ if $r > \delta$. When consumption is postponed to the second period, savings are increasing: r is the return on savings, while δ is the cost of savings (note that consumers prefer consumption today rather than tomorrow). Thus consumption grows over time when the return on savings is larger than the cost of savings.

In contrast with the Solow model, now savings are derived from optimal behavior of households and are not a fixed fraction of income. Let us now introduce investment in fixed capital in the model, so to make income variable over time as a result of capital accumulation.

In per capita terms

$$k_{t+1} = x_t - nk_t$$

where x denotes investments (we assume zero depreciation rate of capital) and n the rate of growth of population

Rewrite the budget constraints, assume :

$c_1 = f(k_1) - k_2(1+n)$ (total income minus investments, all per capita)
$c_2 = f(k_2) + k_2$ (income produced in period 2 plus capital brought from period 1)

$$U(f(k_1) - k_2(1+n)) + \beta U(f(k_2) + k_2)$$

$$U'(c_1)(-(1+n)) + \beta U'(c_2)(1 + f'(k_2) = 0$$

$\frac{U'(c_1)}{U'(c_2)} = \frac{\beta(1+f'(k_{t+1})}{1+n} = \frac{f'(k_{t+1})}{(1+n)(1+\delta)}$

which can be approximated by

$\frac{U'(c_1)}{U'(c_2)} = f'(k_{t+1}) - n - \delta$

In steady state

$f'(k^*) = n + \delta$

Note that k^* is the steady state stock of capital that maximizes utility of households and it is the capital associated with the golden rule, in this case called modified golden rule, as it takes into account the inter-temporal preferences of households (parameter δ).

Extending the model to infinite periods does not change the main thrust of the above results.

Consider two contiguous periods $t, t+1$

$U(c_t) + \beta^t U(c_{t+1})$

$c_t = f(k_t) - k_{t+1}(1+n) + k_t$
$c_{t+1} = f(k_{t+1}) + k_{t+1} - k_{t+2}(1+n)$

Note that the budget constraints can be rewritten in the following way:

$c_t = w_t - I_t + \frac{(1+r)}{1+n} I_{t-1}$
$c_{t+1} = w_{t+1} - I_{t+1} + \frac{(1+r)}{1+n} I_t$
........(the sequence goes on until infinity)

where w is wage income and I investments.

Iterate forward the second budget cosntraint

$c_t = w_t - I_t + \frac{(1+r)}{1+n} I_{t-1}$
$c_{t+1} = w_{t+1} - I_{t+1} + \frac{(1+r)}{1+n} I_t$
$c_{t+2} = w_{t+2} - I_{t+2} + \frac{(1+r)}{1+n} I_{t+1}$

$I_{t+1} = -\frac{(1+n)}{1+r} w_{t+2} + \frac{(1+n)}{1+r} I_{t+2} + \frac{(1+n)}{1+r} c_{t+2}$
plug in c_{t+1}

$c_{t+1} = w_{t+1} + \frac{(1+n)}{1+r} w_{t+2} - \frac{(1+n)}{1+r} I_{t+2} - \frac{(1+n)}{1+r} c_{t+2} + \frac{(1+r)}{1+n} I_t$

$I_t = \frac{(1+n)}{1+r} c_{t+1} - \frac{(1+n)}{1+r} w_{t+1} - \left[\frac{(1+n)}{1+r}\right]^2 w_{t+2} + \left[\frac{(1+n)}{1+r}\right]^2 I_{t+2} + \left[\frac{(1+n)}{1+r}\right]^2 c_{t+2}$

plug in c_t

$c_t = w_t - \frac{(1+n)}{1+r} c_{t+1} + \frac{(1+n)}{1+r} w_{t+1} + \left[\frac{(1+n)}{1+r}\right]^2 w_{t+2} - \left[\frac{(1+n)}{1+r}\right]^2 I_{t+2} - \left[\frac{(1+n)}{1+r}\right]^2 c_{t+2}$

$+ \frac{(1+r)}{1+n} I_{t-1}$

In general, starting from t=0

$\sum_{t=0}^{\infty} \frac{c_t}{[1+r/1+n]^t} = \sum_{t=0}^{\infty} \frac{w_t}{[1+r/1+n]^t} - \lim_{s\to\infty} \frac{I_s}{[1+r/1+n]^s}$

No Ponzi game condition

$\lim_{s\to\infty} \frac{I_s}{[1+r/1+n]^s} \geq 0$

This states that the present value of households asset holdings cannot be negative (households cannot end their life with debts; this would mean that they have not repaid their debts when alive and thus that their consumption was not constrained by the present value of their income. However, it is optimal for households, unless they are satiated, to avoid finishing their life with positive assets. Thus,

$\lim_{s\to\infty} \frac{I_s}{[1+r/1+n]^s} = 0$

And:

$\sum_{t=0}^{\infty} \frac{c_t}{[1+r/1+n]^t} = \sum_{t=0}^{\infty} \frac{w_t}{[1+r/1+n]^t}$

which is the infinite horizon analog of the inter-temporal budget cosntraint derived for 2 periods.

An optimal solution has to satisfy the following conditions for any generic two consecutive periods $t, t+1$:

$\beta^t U'(c_t) + \beta^{t+1} U(c_{t+1}) \frac{dc_{t+1}}{dc_t} = 0$

or

$\frac{dc_{t+1}}{dc_t} = -\frac{\beta^t U'(c_t)}{\beta^{t+1} U'(c_{t+1})}$

The period by period budget constraints

$c_t = f(k_t) - k_{t+1}(1+n) + k_t$
$c_{t+1} = f(k_{t+1}) + k_{t+1} - k_{t+2}(1+n)$

solve for k_{t+1} in the first bc

$$k_{t+1} = \frac{f(k_t)}{(1+n)} + \frac{k_t}{(1+n)} - \frac{c_t}{(1+n)}$$

plug it in the second bc

$$c_{t+1} = f(\frac{f(k_t)}{(1+n)} + \frac{k_t}{(1+n)} - \frac{c_t}{(1+n)}) + \frac{f(k_t)}{(1+n)} + \frac{k_t}{(1+n)} - \frac{c_t}{(1+n)} - k_{t+2}(1+n)$$

An optimal solution should maximize utility and leave unchanged the future budget constraints. In other words, for a given k_t

$$\partial k_{t+2}/\partial c_t = 0$$

Take the total differential of the last bc with respect to a change in c_t :

$$\frac{dc_{t+1}}{dc_t} = f'(k_{t+1})(-\frac{1}{1+n}) - \frac{1}{(1+n)} - \frac{dk_{t+2}(1+n)}{dc_t}$$

$$-\frac{\beta^t U'(c_t)}{\beta^{t+1} U'(c_{t+1})} = -\frac{1}{1+n}\left(f'(k_{t+1}) + 1\right)$$

$$\frac{U'(c_t)}{\beta^t U'(c_{t+1})} = \frac{1}{1+n}\left(f'(k_{t+1}) + 1\right)$$

$$\frac{U'(c_t)}{U'(c_{t+1})} = \frac{\beta}{1+n}\left(f'(k_{t+1}) + 1\right) = \frac{1+f'(k_{t+1})}{(1+n)(1+\delta)}$$

that again can be approximated by

$$\frac{U'(c_t)}{U'(c_{t+1})} = f'(k_{t+1}) - n - \delta$$

which in implicit form defines the so-called Euler equation for optimal consumption dynamics (intuition is identical to the 2-period case). In steady state $c_{t+1} = c_t$ and thus

$$f'(k) = n + \delta$$

9.1 Overlapping generations model

People live for 2 periods, work when young.

$$U(c_t^y) + \beta U(c_{t+1}^o)$$

budget constraints

$c_t^y = w_t - s$
$c_{t+1}^o = s(1+r)$

$\frac{U'(c_t)}{U'(c_{t+1})} = \frac{1+r}{1+\delta}$

Example with logarithmic utility

$\frac{c_{t+1}}{c_t} = \frac{1+r}{1+\delta}$

$c_{t+1}^o = s(1+r)$
$c_t^y \frac{1+r}{1+\delta} = s(1+r)$

$c_t^y = s(1+\delta) = (1+\delta)(w_t - c_t^y)$
$c_t^y = \frac{w_t}{2+\delta}$

Thus savings are

$\frac{w_t}{2+\delta} = w_t - s$
$s = w_t(1 - \frac{w_t}{2+\delta}) = w_t \frac{1+\delta}{2+\delta}$

Individual optimization problem the same as before. Aggregate equilibrium is however different. In each period there are two different groups of consumers, one young, one old.

In equilibrium

$s_t N_t - K_t = K_{t+1} - K_t$

(where we have assumed that capital depreciates completely in one period)

in per capita terms

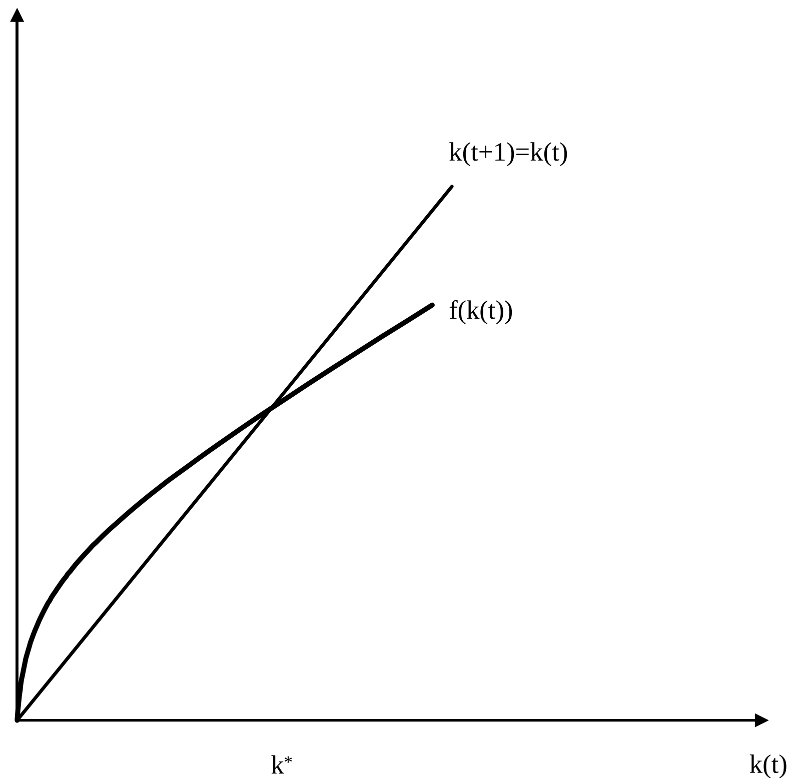
$s_t = k_{t+1}(1+n)$

$k_{t+1}(1+n) = w_t \frac{1+\delta}{2+\delta}$

Profit max by firms implies

$w_t = f(k_t) - k_t f'(k_t)$

$k_{t+1}(1+n) = [f(k_t) - k_t f'(k_t)] \frac{1+\delta}{2+\delta}$



With Cobb-Douglas production function

$[f(k_t) - k_t f'(k_t)] = k_t^\alpha - \alpha k_t k_t^{\alpha-1} = (1-\alpha)k_t^\alpha$
$k_{t+1} = (1-\alpha)k_t^\alpha \frac{1+\delta}{2+\delta} \frac{1}{(1+n)}$

Dynamic equation.

Steady state $k_{t+1} = k_t$

$k^* = \left[(1-\alpha)\frac{1+\delta}{2+\delta} \frac{1}{(1+n)}\right]^{1/(1-\alpha)}$

With $\alpha < 1$, the steady state is stable (see graphical analysis)

10 Growth Accounting

$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$

$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} + (1-\alpha)\frac{\dot{A}}{A}$

The share of labor income on total income is roughly constant at around 0.7 in many industrial countries, thus $\alpha = 0.3$.

With this assumption one can compute the role of Total factor productivity (TFP) and that of factor accumulation in affecting growth of output. Define TFP $B = A^{1-\alpha}$

$\frac{\dot{B}}{B} = (1-\alpha)\frac{\dot{A}}{A}$

$\frac{\dot{Y}}{Y} = 0.3\frac{\dot{K}}{K} + 0.7\frac{\dot{L}}{L} + \frac{\dot{B}}{B}$

or TFP growth

$\frac{\dot{B}}{B} = \frac{\dot{Y}}{Y} - 0.3\frac{\dot{K}}{K} - 0.7\frac{\dot{L}}{L}$

TFP is thus derived as a residual (that is why it is called the Solow residual): net investment and rate of change of employment can indeed be measured. The key assumption in the decomposition is the assumption on α.

In per capita terms

$\frac{Y_t}{L_t} = \frac{K_t^\alpha (A_t L_t)^{1-\alpha}}{L_t} = \frac{K_t^\alpha}{Y_t^\alpha}\frac{Y_t^\alpha}{L_t^\alpha}\frac{(A_t L_t)^{1-\alpha}}{L_t^{1-\alpha}} = \frac{K_t^\alpha}{Y_t^\alpha}\frac{Y_t^\alpha}{L_t^\alpha}\left(\frac{A_t}{L_t}\right)^{1-\alpha}$

take logs

$\ln \frac{Y}{L} = \alpha \ln \frac{K}{Y} + \alpha \ln \frac{Y}{L} + (1-\alpha)\ln \frac{A}{L}$

$(1-\alpha)\ln \frac{Y}{L} = \alpha \ln \frac{K}{Y} + (1-\alpha)\ln \frac{A}{L}$

$\ln \frac{Y}{L} = \frac{\alpha}{(1-\alpha)}\ln \frac{K}{Y} + \ln \frac{A}{L}$

When we consider the production function with human capital as well

$\frac{Y_t}{L_t} = \frac{K_t^\alpha (A_t H_t)^{1-\alpha}}{L_t} = \frac{K_t^\alpha}{Y_t^\alpha}\frac{Y_t^\alpha}{L_t^\alpha}\frac{(A_t H_t)^{1-\alpha}}{L_t^{1-\alpha}} = \frac{K_t^\alpha}{Y_t^\alpha}\frac{Y_t^\alpha}{L_t^\alpha}\left(\frac{H_t}{L_t}\right)^{1-\alpha} A_t^{1-\alpha}$

take logs

$\ln \frac{Y}{L} = \alpha \ln \frac{K}{Y} + \alpha \ln \frac{Y}{L} + (1-\alpha) \ln \frac{H}{L} + (1-\alpha) \ln A$

$(1-\alpha) \ln \frac{Y}{L} = \alpha \ln \frac{K}{Y} + (1-\alpha) \ln \frac{H}{L} + (1-\alpha) \ln A$

$\ln \frac{Y}{L} = \frac{\alpha}{1-\alpha} \ln \frac{K}{Y} + \ln \frac{H}{L} + \ln A$

This is a useful decomposition for empirical analysis at the cross-country level or for long time series for a specific country. Again $\ln A$ is a residual, as capital/output ratio and H/L, for instance years of schooling divided by population can be measured. One could thus identify the role of TFP in explaining diffeent levels of incomes per capita, or productivity.

11 A unified theory of growth and development

Recent literature has tried to organize in a unified framewrok the analysis of economic growth over a very long span of time. The focus has been on the institutional and policy factors that determine the different performance of countries and the time when a country starts a period of rapid growth of incomes per capita (main readings: Ngai(2004), Parente and Prescott (2004))

A pre-industrial revolution period is defined the Malthus regime, one in which incomes grow proportionally with population and thus incomes per capita are roughly constant over long periods of time. The regime of sustained growth of incomes per capita is defined as the Solow regime.

11.1 Two setor model of growth and development

Malthusian production function

Production with three factors, of which one is fixed (land)

Technology: Cobb-Douglas

$Y_{Mt} = A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu} L_M^{1-\phi-\mu}$

Total factor productivity:

$A_{Mt} = A_{M0} {\gamma_M}^t$

Resource constraint, indicates that output can be used for consumption or investment

$C_t + I_t = Y_{Mt}$

Solow production function

$Y_{st} = A_{st} K_{st}^{\theta} N_{st}^{1-\theta}$
Preferences (Overlapping Generations)

People live two periods (first period, they are young, second they are old). They supply inelastically 1 unit of labor when they are young. In the first period

they can consume or invest in capital or land. When they are old they consume all their income from investments. Capital fully depreciates in one period.

utility= $\log c_{1t} + \beta \log(c_{2t+1})$

budget constraints

$c_{1t} = w_t - k_{M,t+1} - k_{s,t+1} - q_t l_{t+1}$
$c_{2,t+1} = r_{kM,t+1} \frac{k_{M,t+1}}{\pi_M} + r_{ks,t+1} \frac{k_{s,t+1}}{\pi_s} + (r_{l,t+1} + q_{t+1}) l_{t+1}$

where r indicates rates of return of the three assets (capital in the Malthus technology, capital in the Solow technology and land); q is the price of land, that can vary to equalize demand and (fixed) supply. π denotes the barrier to the efficient use of capital in the two technologies. $\pi = 1$ occurs when there are no barriers to the efficient use of capital. In equilibrium, the real return on the various assets has to be equal in order to ensure that the consumer has no incentive to change its allocation of savings across the three assets

Thus:

$\frac{r_{kM,t+1}}{\pi_M} = \frac{r_{ks,t+1}}{\pi_s} = R_{t+1} = \frac{q_{t+1}}{q_t} + \frac{r_{l,t+1}}{q_t}$

where the latter expression indicates the real return on land inclusive of the capital gain obtained because of an increase in the price of land.

We can rewrite the budget constraints as follows

$c_{1t} = w_t - k_{t+1} - q_t l_{t+1}$
$c_{2t+1} = R_{t+1} k_{t+1} + (r_{l,t+1} + q_{t+1}) l_{t+1}$

Divide the last equation by R_{t+1}

$\frac{c_{2t+1}}{R_{t+1}} - \frac{(r_{l,t+1} + q_{t+1}) l_{t+1}}{R_{t+1}} = k_{t+1}$

and substitute this value for k_{t+1} in the period 1 budget constraint, to obtain the intertemporal budget constraint

$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} = w_t + \frac{(r_{l,t+1}+q_{t+1})l_{t+1}}{R_{t+1}} - q_t l_{t+1}$

or

$c_{1t} + \frac{c_{2t+1}}{R_{t+1}} = w_t + \left(\frac{r_{l,t+1}+q_{t+1}}{R_{t+1}} - q_t\right) l_{t+1}$

Maximization by representative consumer: Write the Lagrangean

L=$\log c_{1t} + \beta \log(c_{2t+1}) - \lambda \left[c_{1t} + \frac{c_{2t+1}}{R_{t+1}} - w_t + \left(\frac{r_{l,t+1}+q_{t+1}}{R_{t+1}} - q_t\right) l_{t+1}\right]$

$\partial L/\partial c_{1t} = 0 = \partial L/\partial c_{2t+1}$

$\frac{1}{c_{1t}} - \lambda = 0$
$\frac{\beta}{c_{2t+1}} - \lambda \frac{1}{R_{t+1}} = 0$

Eliminating λ
$\frac{\beta R_{t+1}}{c_{2t+1}} = \frac{1}{c_{1t}}$

or

$\frac{\beta R_{t+1}}{c_{2t+1}} = \frac{1}{c_{1t}}$

$\frac{c_{2t+1}}{R_{t+1}} = \beta c_{1t}$

substitute in the intertemporal budget constraint

$c_{1t} + \beta c_{1t} = w_t + (\frac{q_{t+1}}{R_{t+1}} - q_t + \frac{r_{l,t+1}}{R_{t+1}})l_{t+1}$

in equilibrium (equality between real return on different assets)

$R_{t+1} = \frac{r_{l,t+1}}{q_t} + \frac{q_{t+1}}{q_t}$
$q_t = \frac{r_{l,t+1}+q_{t+1}}{R_{t+1}}$
$\frac{q_{t+1}}{R_{t+1}} - q_t + \frac{r_{l,t+1}}{R_{t+1}} = 0$

and thus

$c_{1t} = \frac{w_t}{1+\beta}$

In aggregate, the equilibrium respects the condition

$c_{1t} + \frac{K_{t+1}}{N_t} + q_t \frac{1}{N_t} = w_t$

or

$K_{t+1} = (w_t - c_{1t})N_t - q_t$

11.2 Optimal behavior by firms

11.2.1 Malthus sector

Profits

$\Pi_{Mt} = A_{Mt}K_{Mt}^{\phi}N_{Mt}^{\mu}L^{1-\mu-\phi} - r_{k,Mt}K_{Mt} - w_t N_{Mt} - r_{lt}L$

choose K and N to max Π_{Mt}

$\partial \Pi_{Mt}/\partial K = 0$

$\phi A_{Mt}K_{Mt}^{\phi-1}N_{Mt}^{\mu}L^{1-\mu-\phi} = r_{k,Mt}$

$\partial \Pi_{Mt}/\partial N = 0$

$\mu A_{Mt}K_{Mt}^{\phi}N_{Mt}^{\mu-1}L^{1-\mu-\phi} = w_t$

combine the two conditions (divide the first by the second)

$\frac{\phi}{\mu}\frac{N}{K} = \frac{r}{w}$

or

$K = \frac{\phi}{\mu}\frac{w}{r}N$

plug it in the second condition, using the fact that $L = 1$

$N = A_{Mt}^{1/(1-\mu-\phi)} \left(\frac{\mu}{w}\right)^{(1-\phi)/(1-\mu-\phi)} \left(\frac{\phi}{r}\right)^{\phi/(1-\mu-\phi)}$
$K = A_{Mt}^{1/(1-\mu-\phi)} \left(\frac{\mu}{w}\right)^{\mu/(1-\mu-\phi)} \left(\frac{\phi}{r}\right)^{(1-\mu)/(1-\mu-\phi)}$

Thus output and profits can be rewritten in terms of w, r

$\Pi_{Mt} = A_{Mt}^{1/(1-\mu-\phi)}(1-\mu-\phi) \left(\frac{\mu}{w}\right)^{\mu/(1-\mu-\phi)} \left(\frac{\phi}{r}\right)^{\phi/(1-\mu-\phi)} > 0$

Thus, the Malthus technology will always be implemented

There exists a balanced growth path along which per capita output is constant. This is obtained in connection with
$g(c_{1t}) = \gamma^{1/(1-\mu-\phi)}$

Let us compute per capita output:

$Y_{Mt} = A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu} L^{1-\mu-\phi}$

multiply and divide by Y_{Mt}^{ϕ}

$Y_{Mt} = A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu} L^{1-\mu-\phi} \frac{Y_{Mt}^{\phi}}{Y_{Mt}^{\phi}}$

define with $v_{1t} = K_{Mt}/Y_{Mt}$ with $\pi_M = 1$

i.e. the capital-output ratio corresponding to the efficient use of capital

$Y_{Mt} = A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu} L^{1-\mu-\phi} \frac{Y_{Mt}^{\phi}}{Y_{Mt}^{\phi}}$
$Y_{Mt}^{1-\phi} = A_{Mt} \left(\frac{K_{Mt}}{Y_{Mt}}\right)^{\phi} N_{Mt}^{\mu} L^{1-\mu-\phi} = A_{Mt} \left(\frac{v_{1t}}{\pi_M}\right)^{\phi} N_{Mt}^{\mu} L^{1-\mu-\phi}$

Divide both sides by $N_t^{1-\phi}$

$$\widehat{y}_{Mt} = \left[A_{Mt} \left(\frac{K_{Mt}}{Y_{Mt}}\right)^{\phi} \left(\frac{L}{N_{Mt}}\right)^{1-\mu-\phi}\right]^{1/(1-\phi)}$$

note that $A_{Mt} = A_M \gamma_{Mt}^t$

$$\widehat{y}_{Mt} = \left[A_M \gamma_{Mt}^t \left(\frac{K_{Mt}}{Y_{Mt}}\right)^{\phi} \left(\frac{L}{N_{Mt}}\right)^{1-\mu-\phi}\right]^{1/(1-\phi)}$$

along the balanced growth path population growth is $\gamma^{1/(1-\mu-\phi)}$

thus $N_t = N_o \left(\gamma^{1/(1-\mu-\phi)}\right)^t$

$$\widehat{y}_{Mt} = \left[A_M \gamma_{Mt}^t \left(\frac{K_{Mt}}{Y_{Mt}}\right)^{\phi} \left(\frac{L}{N_o\left(\gamma^{1/(1-\mu-\phi)}\right)^t}\right)^{1-\mu-\phi}\right]^{1/(1-\phi)} = \left[A_M \left(\frac{v_{1t}}{\pi_M}\right)^{\phi} \left(\frac{L}{N_o}\right)^{1-\mu-\phi}\right]^{1/(1-\phi)}$$

Thus per capita output is constant and it is independent of the rate of growth of Total factor productivity (γ)

11.2.2 Solow Technology

The Solow sector operates when incomes, and consumption, per capita have reached a sufficiently high level that population growth becomes independent of living standards.

$Y_{st} = A_{st} K_{st}^{\theta} N_{st}^{1-\theta}$

optimal choice of K and N

max profits

$r_{k,st} = \theta A_{st} K_{st}^{\theta-1} N_{st}^{1-\theta}$
$w_t = (1-\theta) A_{st} K_{st}^{\theta} N_{st}^{-\theta}$

from which

$\frac{K}{N} = \frac{w}{r} \frac{\theta}{1-\theta}$

Define per capita profits as

$\frac{\Pi_{st}}{N_t} = A_{st} (\frac{K_t}{N_t})^{\theta} - w_t - r_t \frac{K_t}{N_t} = A_{st} (\frac{w_t}{r_t} \frac{\theta}{1-\theta})^{\theta} - w_t - r_t \frac{w_t}{r_t} \frac{\theta}{1-\theta} = A_{st} (\frac{w_t}{r_t} \frac{\theta}{1-\theta})^{\theta} - w_t \frac{1}{1-\theta} > 0$

if

$A_{st} (\frac{w_t}{r_t} \frac{\theta}{1-\theta})^{\theta} > w_t \frac{1}{1-\theta}$

$A_{st} > \left(\frac{w_t}{1-\theta}\right)^{1-\theta} \left(\frac{r_t}{\theta}\right)^{\theta}$ (*)

Solow technology adopted only if TFP growth is sufficiently high.

Income per capita in the Solow sector increases at the rate of growth of TFP

$Y_{st} = A_s \gamma_s^t K_{st}^{\theta} N_{st}^{1-\theta}$

Multiply and divide the above expression by Y_{st}^{θ}

$Y_{st}^{1-\theta} = A_s \gamma_s^t \left(\frac{K_{st}}{Y_{st}}\right)^{\theta} N_{st}^{1-\theta} = A_s \gamma_s^t \left(\frac{v_{1s}}{\pi_s}\right)^{\theta} N_{st}^{1-\theta}$

divide both sides by $N_{st}^{1-\theta}$

$\frac{Y_{st}^{1-\theta}}{N_{st}^{1-\theta}} = A_s \gamma_s^t \left(\frac{K_{st}}{Y_{st}}\right)^{\theta} N_{st}^{1-\theta} = A_s \gamma_s^t \left(\frac{v_{1s}}{\pi_s}\right)^{\theta}$

Income per capita along the balanced growth path of the Solow sector

$$\widehat{y}_{st} = \left[A_s \gamma_s^t \left(\frac{v_{1s}}{\pi_s}\right)^{\theta}\right]^{1/(1-\theta)}$$

Per capita incomes grow at the rate of growth of TFP.

11.3 Development and transition

Development can be seen as a transition from Malthus to Solow:

Assume the economy is intially on the balanced growth path with only the Malthus secor operating. When it becomes profitable to adopt the Solow technology, the economy operates with both sectors. At that point, assuming free labor mobility, and free capital mobility across sectors, there are equilibrium conditions to be satisfied. Recall that there are barriers to the efficient use of capital, π_M in the Malthus sector, π_s in the Solow sector. In equilibrium, wages are equal for workers emplyed in both sectors, and the return on capital also has to be equal. Thus,

$$\frac{r_{st}}{\pi s} = \frac{r_{Mt}}{\pi_M}$$

Thus, at the point of adoption of Solow technology, (*) can be rewritten as

$$A_{st} > \left(\frac{w_{Mt}}{1-\theta}\right)^{1-\theta} \left(\frac{r_{Mt}}{\theta}\frac{\pi_s}{\pi_M}\right)^{\theta}$$

reacall that w_{Mt} and r_{Mt} are independent of the rate of growth of TFP in the Malthus secotor, and depend only on the capital/labor ratio, the per capita land at time 0 and other preference and technology parametrs.

Define with t^* the turning point to the Solow technolgy, that occurs when inequality (*) is satisfied for the first time

$$A_s \gamma_s^{t^*} \geqslant B \pi_s^{\theta} \pi_M^{-\phi(1-\theta)/(1-\phi)} (L/N_0)^{(1-\mu-\phi)(1-\theta)/(1-\phi)} > A_s \gamma_s^{t^*-1} \qquad (**)$$

The condition above is obtained by inserting the values of w and r corresponding to the balanced growth path in the economy with only the Malthus sector.

where $B = \left(\frac{\phi}{\theta}\right)^{\theta} \left(\frac{\mu}{1-\theta}\right)^{1-\theta} \left[v_{1M}^{\phi-\theta} A_M^{1-\theta}\right]^{1/(1-\phi)}$

Note that the right-hand-side of (**) is constant along a balanced growth path, while the left-hand-side is increasing through time: therefore, eventually an economy will begin its transition, by adopting the Solow technology. The key issue is when such a transition will start.

11.4 Barriers to development

Condition (**) illustrates the factors that can induce delay in the transition to a dynamic Solow-type economy. From (**) one can compute the time of transition. Indeed, taking logs of (**)

$t^* \log A_s\gamma_s \geqslant \log B + \theta \log \pi_s - \phi(1-\theta)/(1-\phi)\pi_M +^{(1-\mu-\phi)(1-\theta)/(1-\phi)} \log(L/N_0)$ (***)

the critical turning point is

$t^* = \frac{1}{\log A_s\gamma_s}\left[\log B + \theta \log \pi_s - \phi(1-\theta)/(1-\phi)\pi_M +^{(1-\mu-\phi)(1-\theta)/(1-\phi)} \log(L/N_0)\right]$ (****)

Different starting points of transition are explained by the factors on the right-hand-side of condition (****).

The interesting point is that those factors are crucially related to institutional and policy factors.

Note that the Malthus sector is not necessarily the agricultural sector, although it coincides in history with it. It represents all sectors in which productivity does not increase. In the studies of transition from socialism to a market economy, the Malthus sector has been considered the state sector, while the Solow sector the new private sector (see Chadha and Coricelli (1995) and Campos and Coricelli (2002) for the discussion of growth in transition countries).

11.5 Dynamics of allocation of labor across sectors

Note that as the Solow sector is characterized by a continuous growth in TFP, it will attract workers out of the Malthus sector: eventually all workers will move to the Solow sector.

F.O.C. in Malthus sector
$\partial\Pi_{Mt}/\partial K = \phi\frac{Y_M}{K_M} = r_m$
$\partial\Pi_{Mt}/\partial N = \mu\frac{Y_M}{N_M} = w_m$
$\partial\Pi_{Mt}/\partial L = (1-\mu-\phi)\frac{Y_M}{L} = r_L$

F.O.C. in Solow sector
$\partial\Pi_{st}/\partial K = \theta\frac{Ys}{Ks} = r_s$
$\partial\Pi_{st}/\partial N = (1-\theta)\frac{Ys}{Ns} = w_s$

From equalization of real returns on fixed investments and wages across sectors

(equal return on capital)
$\frac{\theta Y_{st}}{\pi_s K_{s,t}} = \phi \frac{Y_{Mt}}{\pi_M K_{Mt}}$

(equal wages)

$\frac{(1-\theta)Y_{st}}{N_{s,t}} = \frac{\mu Y_{Mt}}{N_{Mt}}$

Combining the two identities

$k_{Mt} = k_{st} \frac{\phi(1-\theta)}{\mu\theta} \frac{\pi_s}{\pi_M}$

Define $\Psi = \frac{\phi(1-\theta)}{\mu\theta}$

$k_{Mt} = k_{st} \Psi \frac{\pi_s}{\pi_M}$

From market clearing (total capital stock in t+1 equals investment in period t)

$$I_t = (w_{t-1} - c_{1t-1})N_{t-1} - q_{t-1}$$
$$\pi_M k_{Mt} N_{Mt} + \pi_s k_{st} N_{st} = I_t$$
$$N_{Mt} = N_t n_M$$
$$N_{st} = N_t(1 - n_M)$$
$$\pi_M k_{Mt} N_t n_M + \pi_s k_{st} N_t (1 - n_M) = I_t$$

$$\pi_M k_{Mt} n_M + \frac{1}{\Psi} \pi_M k_{Mt} (1 - n_M) = \frac{I_t}{N_t}$$
$$\pi_M k_{Mt} \left(\frac{1-(1-\Psi)n_M}{\Psi} \right) = \frac{I_t}{N_t}$$
$$k_{Mt} = \frac{\Psi}{[1-(1-\Psi)n_{Mt}]\pi_M} \frac{I_t}{N_t} \qquad \text{(MCC)}$$

In equilibrium workers have to be indifferent between being employed in the Malthus or in the Solow sector:

$$w_{st} = w_{Mt}$$

$$w_{st} = (1-\theta) A_{st} K_{st}^{\theta} N_{st}^{-\theta}$$
$$w_{Mt} = \mu A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu-1} L^{1-\mu-\phi}$$

$$(1-\theta) A_{st} K_{st}^{\theta} N_{st}^{-\theta} = \mu A_{Mt} K_{Mt}^{\phi} N_{Mt}^{\mu-1} L^{1-\mu-\phi}$$

use

$$k_{Mt} = k_{st} \Psi \frac{\pi_s}{\pi_M}$$

to get

$$k_{Mt}^{\theta-\phi} = \frac{\mu}{1-\theta}\frac{A_{Mt}}{A_{st}}\left(\Psi\frac{\pi_s}{\pi_M}\right)^{\theta}\left(\frac{L}{N_{Mt}}\right)^{1-\mu-\phi} \qquad \text{(labor indifference)}$$

Combining MCC with the condition for labor indifference, we obtain

$$f(n_{Mt}^*) = \frac{\mu}{1-\theta}\pi_s^{\theta}\pi_M^{-\phi}\left[1-(1-\Psi)n_{Mt}\right]^{\theta-\phi} - \frac{A_{Mt}}{A_{st}}I_t^{\theta-\phi}N_t^{1-\theta-\mu}n_{Mt}^{1-\mu-\phi}$$

n_{Mt}^* is the equilibrium share of labor in the Malthus sector and thus solves

$$f(n_{Mt}^*) = 0$$

n_{Mt}^* exists and is unique as $f(n_{Mt}^*)$ is monotonically decreasing in n_{Mt}^* (excercise: show that $f'(n_{Mt}^*) < 0$)

$f(0) > 0;\ \ f(1) < 0$

Notice that the second element of $f(n_{Mt}^*)$, which is negative, grows continuosly over time if $\gamma_s > \gamma_M$, while the first element does not grow continuously over time. Thus eventually the second, negative, element will become larger in absolute value and thus $f(n_{Mt}^*)$ will be smaller and smaller, converging to zero for n_{Mt}^* going to zero. Thus, eventually, all workers will be employed in the Solow sector. The issue is how fast such process lasts (see Chadha and Coricelli (1994) for an application to transition from planned to market economies).

12 Transition model

(B. Chadha and F. Coricelli (1995), "Unemployment, investment and sectoral reallocation", CEPR Discussion Papers)

The labor market

Rural (Malthus) sector

Output

$$Q_{1t} = H_1(L_{1t})^{\beta}$$

with $\quad 0 < \beta < 1$

H is constant, and the fixed factor, land, has been suppressed for easyness of exposition

Labor can move freely across sectors. Workers in the rural sector can thus freely seek jobs in the urban (Solow) sector. In equilibrium, wages in the rural sector (equal to the marginal product of labor in that sector, equal the expected income from leaving the sector (taking into account that workers can remain unemployed.

$$\beta(L_{1t})^{\beta-1} = \frac{1}{P}\left[\delta_t B + (1-\delta_t)W_{2t}\right]$$

where δ_t denotes the probability that a worker leaving the rural sector becomes unemployed. B denotes the exogenous level of unemployment benefits paid by the government, W_2 is the wage in the urban sector and P is the internationally given relative price of the rural sector good in terms of the urban sector good.

Urban (Solow) sector

We assume that effort by workers in the urban sector is an increasing, concave function of the premium of real wages over unemployment benefits, and of the level of unemployment.

$$Q_{2t} = H_2\left[E(W_{2t}-B, U_t)L_{2t}\right]^{\alpha} K_t^{1-\alpha}$$

with H_2 constant and $0 < \alpha < 1$

$E(0, U_t) < 0$ for all U_t

Efficiency condition:

$$\frac{\partial E(W_{2t}-B,U_t)}{\partial W_{2t}}\frac{W_{2t}}{E(W_{2t}-B,U_t)} = 1$$

which implies that

$$W_{2t} = W_2(U_t)$$

with

$$\frac{\partial W_2(U_t)}{\partial U} < 0$$

$$L_{2t} = \frac{(\alpha H_2)^{1/(1-\alpha)}[E(W_{2t}-B,U_t)]^{\alpha/(1-\alpha)}K_t}{[W_{2t}]^{1/(1-\alpha)}} = L_2\left[W_2(U_t), U_t, K_t\right] = L_2(U_t, K_t)$$

Equilibrium Unemployment

$1-\delta_t = \frac{L_{2t}}{U_t + L_{2t}}$

inserting this condition in the equation that eqautes wages in the rural sector with expected income from leaving that sector, and assuming that actual and expected values coincide, we obtain

$L_{1t} = L_{1t}\left[U_t, W_2(U_t), L_{2,}U_t, K_t); P\right]$

or
$L_{1t} = L_1(U_t, K_t; P)$

with

$\frac{\partial L_1}{\partial U} > 0$

$\frac{\partial L_1}{\partial K} < 0$

$\frac{\partial L_1}{\partial P} > 0$

In general equilibrium

$L_1(U_t, K_t; P) + L_2(U_t, K_t) + U_t = 1$

which yields the equilibrium level of unemployment in terms of the stock of the capital stock.

The left-hand side of the equilibrium condition is unambiguously increasing in unemployment, but an ambiguous function of the capital stock.

Figure 1: Nonlinear (hump-shaped) relation between unemployment and the capital stock

Investment in the Urban sector

$\frac{\partial Q_{2t}}{\partial K_t} = (1-\alpha)H_2\left[E(W_{2t} - B, U_t)L_{2t}\right]^{\alpha} K_t^{-\alpha}$

Plug the optimal value of L_2 to obtain:

$$\left[\frac{\partial Q_{2t}}{\partial K_t}\right]_{L_2 = L_2^*} = (1-\alpha)H_2\left[E(W_{2t} - B, U_t)L_{2t}\right]^{\alpha} K_t^{-\alpha} = (1-\alpha)H_2\Big[E(W_{2t} - B, U_t)\frac{(\alpha H_2)^{1/(1-\alpha)}[E(W_{2t}-B,U_t)]^{\alpha/}}{[W_{2t}]^{1/(1-\alpha)}}$$

$$\left[\frac{\partial Q_{2t}}{\partial K_t}\right]_{L_2 = L_2^*} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} H_2^{\frac{1}{1-\alpha}} \left[\frac{E(W_{2t}(U_t) - B, U_t)}{W_{2t}(U_t)}\right]^{\frac{\alpha}{1-\alpha}} = \theta G(U_t)$$

Around an equilibrium point (optimal wage)

$$\frac{\partial G}{\partial U} = \theta \frac{\alpha}{1-\alpha} \left[\frac{E(W_{2t}-B, U_t)}{W_{2t}} \right]^{\frac{\alpha}{1-\alpha}} \frac{1}{E_t} \frac{\partial E}{\partial U_t} > 0$$

and thus

$$\frac{\partial Q_{2t}}{\partial K_t} = \theta G(U_t) > 0$$

The marginal product of labor mirrors the behavior of the hump-shaped curve, relating unemployment to the capital stock

$$\frac{\partial Q_{2t}}{\partial K_t} = \theta G(U(K_t))$$

Dynamics of the capital stock

Myopic investors: investments based only on current returns:

$$\dot{K}_t = \gamma(r_t - r^*)$$

where r^* is the exogenously given world real rate of interest

$$\dot{K}_t = \gamma(\theta G(U_t) - r^*)$$

$$\dot{K}_t = 0$$

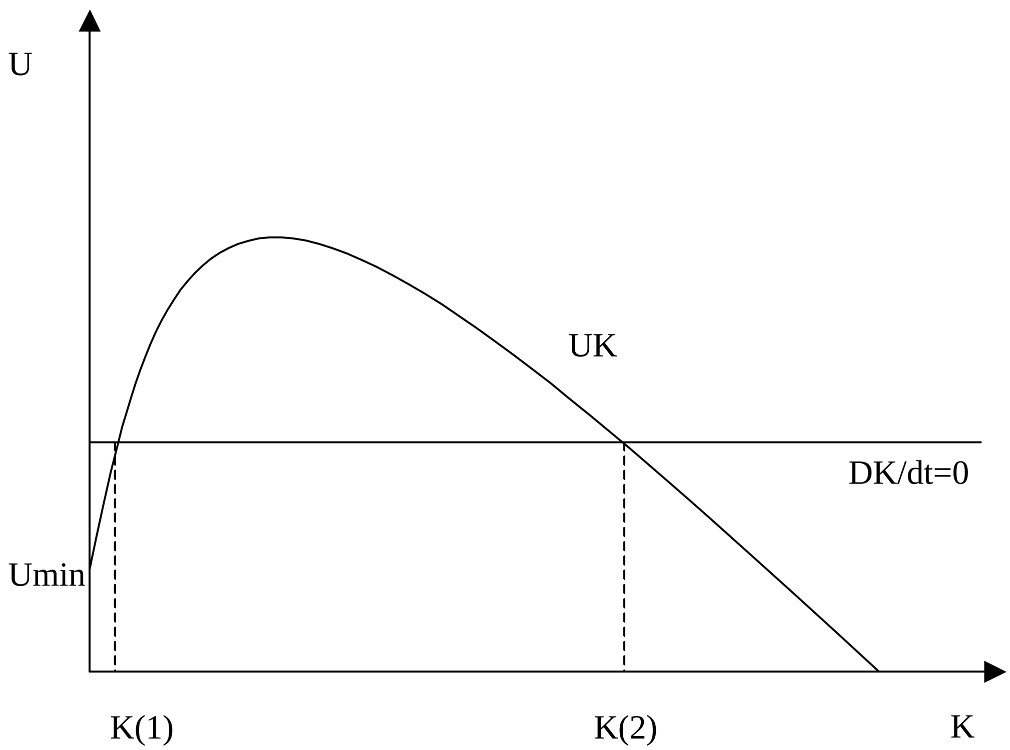
U^* solves

$$\theta G(U^*) = r^*$$

Note that the economy is always on the UK curve. We can define three types of equilibria:

1. Failed transition; the industrial sector disappears, all workers will go to the rural sector
2. Low output equilibrium; the system gets stuck at a low level of capital stock and at a low level of employment in the urban sector
3. Successful transition: high level of capital stock in the industrial sector, a large amount of workers shift from rural to urban sector

As there is no trend growth of TFP in the urban sector not all workers will eventually move to the urban sector.


U
UK
DK/dt=0
Umin
K(1)
K(2)
K

13 The role of policies and institutions

This part will be non-technical and largely empirical.

See Romer and Parente-Prescott, 2004; On the role of policies on competition, market regulation and their effects on growth see OECD, Nicoletti and Scarpetta (2005)